J. Fluid Mech. (1993), *vol.* 252, *pp.* 565–584

Surface waves on shear currents: solution of the boundary-value problem

By VICTOR I. SHRIRA

P. P. Shirshov Institute for Oceanology Russian Academy of Sciences, Krasikova 23, Moscow 117218, Russia

(Received 15 July 1991 and in revised form 31 July 1992)

We consider a classic boundary-value problem for deep-water gravity–capillary waves in a shear flow, composed of the Rayleigh equation and the standard linearized kinematic and dynamic inviscid boundary conditions at the free surface. We derived the exact solution for this problem in terms of an infinite series in powers of a certain parameter ϵ, which characterizes the smallness of the deviation of the wave motion from the potential motion. For the existence and absolute convergence of the solution it is sufficient that ϵ be less than unity.

The truncated sums of the series provide approximate solutions with *a priori* prescribed accuracy. In particular, for the short-wave instability, which can be interpreted as the Miles critical-layer-type instability, the explicit approximate expressions for the growth rates are derived. The growth rates in a certain (very narrow) range of scales can exceed the Miles increments caused by the wind.

The effect of thin boundary layers on the dispersion relation was also investigated using an asymptotic procedure based on the smallness of the product of the layer thickness and wavenumber. The criterion specifying when and with what accuracy the boundary-layer influence can be neglected has been derived.

1. Introduction

Water waves in oceans and other natural basins almost always propagate on shear currents, rather than in still water. The theory of the interaction between waves and steady currents has many different aspects, among them a number of open ones (see the review works of Peregrine 1976; Peregrine & Jonsson 1983; Jonsson 1989; Craik 1985). In this paper we shall focus our attention upon one of the most fundamental open questions, which is the inevitable first step in any study of wave–current interactions, namely the boundary-value problem.

The boundary-value problem we are interested in follows naturally from the fundamental equations of inviscid hydrodynamics if one considers small monochromatic perturbations to the steady horizontally uniform shear flow $U(z)$. It implicitly gives the vertical structure $W(z)$ of monochromatic perturbations with wave vector $\boldsymbol{k}$, and their phase velocity C in terms of the basic flow profile and the wave vector.

The problem reduces under certain additional assumptions discussed in §2.1 to the boundary-value problem prescribed by the Rayleigh equation

$$(C-\mathscr{U})(W''-k^2W)+\mathscr{U}''W=0, \tag{1.1}$$

zero boundary condition at the lower horizontal boundary $z=-H$,

$$W=0, \tag{1.2}$$

and the standard boundary condition at the free surface ($z = 0$) in the form

$$F_1(k, C, \mathscr{U}(0), \mathscr{U}'(0))\, W + F_2(k, C, \mathscr{U}(0), \mathscr{U}'(0))\, W' = 0, \qquad (1.3)$$

where $\mathscr{U}$ and C are the projections of the current velocity $\boldsymbol{U}$ and the phase velocity $\boldsymbol{C}$ in the $\boldsymbol{k}$-direction, F_1 and F_2 are the known functions of their arguments specified below in §2.1. We do not specify F_1 and F_2 here to emphasize that the boundary-value problem (1.1)–(1.3) also arises in many other physical contexts (e.g. waves in an elastic tube, wave–wind interaction, barotropic instability of Rossby waves on a jet with Rayleigh viscosity taken into account, etc.) and our results derived for water waves can be applied to many other physical problems.

A less general case of this boundary-value problem, namely with F_2 equals to zero, is the classical problem thoroughly analysed within the context of the linear hydrodynamic stability theory in many textbooks and monographs (e.g. Lin 1966; Dikii 1976; Drazin & Reid 1981).

The corpus of works devoted directly to the boundary-value problem (1.1)–(1.3) is also considerable (see references in recent works by Jonsson 1989; Kirby & Chen 1989; Henyey & Wright 1989). Several different approaches have been attempted to attack this problem.

Henyey & Wright (1989), Wright & Henyey (1989) studied general properties of the normal modes decomposition and have proved the completeness of the basis composed of the continuous and discrete spectra. Shrira (1989) studied the evolution of the continuous spectrum when the basic flow vorticity is localized in the narrow (in comparison to a characteristic wavelength) subsurface layer. It has been shown that the continuous spectrum modes can be approximately treated within a certain temporal interval as a single discrete spectrum mode. All other researchers concentrated their attention on the surface mode, which is of evident prime interest in the context of water waves.

The search for analytical solutions proved to be successful for a very limited number of specific current profiles: linear (Thompson 1949; Biesel 1950), exponential (Abdullah 1949), and for $z^{\frac{1}{7}}$ (Lighthill 1953; Fenton 1973) (only solutions corresponding to zero eigenvalue, i.e. $C = 0$, can be found). We note that none of these profiles contains any free parameters to construct satisfactory approximations for the real profiles, but are often useful to test other approaches.

The semi-analytical approach based on the piecewise linear current velocity approximation (e.g. Gertsenshtein, Romashova & Chernyavski, 1988) reduces the problem to an algebraic equation of the order $N+2$, where N is the number of velocity corners (vorticity jumps). Only the simplest non-trivial case ($N = 1$) can be treated analytically. Higher-order approximations inevitably require numerical methods. Many popular program packets for the dispersion curve calculation are based on this approach. The discussion of this approach as well as other pure numerical methods lies far beyond the aims of this work; however, we note that convergency of these solutions to the solution of the corresponding continuous problem, as N tends to infinity, has not been rigorously proved yet.

There is a number of works, similar in spirit, where approximate analytical solutions were constructed exploiting the smallness of the ratio of characteristic current velocity to wave phase velocity. Stewart & Joy (1974) were the first to employ this small parameter and to derive the first-order correction to the phase velocity for the deep-water case. Skop (1987) obtained the next-order term. Kirby & Chen (1989) extended the analysis for finite water depth, calculated the second-order terms and claimed their importance by careful comparison with the several known exact solutions. Still the questions regarding the accuracy and the range of validity of these approximate solutions remain open, however. Obviously for waves with phase velocities of the order of the mean current or smaller a quite new approach is necessary. Kirby & Chen (1989) derived for this range 'weak vorticity' approximate solutions by perturbing the linear velocity case.

In our work we shall concentrate on the derivation of analytical solutions to the boundary-value problem (1.1)–(1.3) relevant to water waves in typical oceanic conditions. In mathematical terms this means the following:

(i) we shall be interested in discrete spectrum solutions;
(ii) we shall exploit the relations among the problem parameters typical for the ocean environment.

Our principal aims are to construct both the exact solutions for the surface mode of the boundary-value problem for an arbitrary shear profile and convenient approximate formulae uniformly valid in the range of ocean wave scales with *a priori* prescribed accuracy.

The main idea of the work can be expressed very simply as follows. Wind waves are commonly believed to be very close to potential ones. Accepting this without criticism as a starting point we exploit this fact to develop a mathematical theory. In mathematical terms this 'near-potentiality' means assured smallness of the last term in the Rayleigh equation (1.1). The non-trivial fact is that perturbation series proved to be convergent (and not only for small deviations from potentiality), which enables us to get exact solutions in the form of an infinite series, while the truncated sums are natural approximate solutions with *a priori* prescribed accuracy.

The second essentially new result of the work, justifying an application of these approximate solutions, quantitatively specifies the conditions for when and with what accuracy the effect of a thin subsurface boundary layer on the solution of boundary-value problem can be neglected.

The paper is organized as follows. Section 2 gives the problem statement. Starting with the set of inviscid hydrodynamics equations we derive, under commonly accepted assumptions, the boundary-value problem (1.1)–(1.3). Considerable attention is also paid in this section to distinguishing the basic non-dimensional parameters and relations among them. The parameter ϵ, which is defined as the ratio of the averaged mean flow vorticity gradient and the product of a characteristic wave vector and wave frequency, is identified as they key non-dimensional parameter of the problem.

In §3, the main part of the work, we construct the exact solution to the boundary-value problem in terms of an absolutely converging series in powers of ϵ, and study some of its properties. We note that for convergence the smallness of ϵ is not necessary. The solution proved to be a Neumann series for the integral presentation of the boundary-value problem. A brief analysis of the approximate solutions given by the leading-order terms in ϵ for different ranges of the other controlling parameters is given in §4.

In §5 the instability which takes place owing to the wave interaction with its critical layer is briefly analysed. The instability of a shear flow with the free surface under the influence of gravity and surface tension has been well studied within the framework of piecewise linear models (e.g. Gertsenstein *et al.* 1988). Within these models the instability can be interpreted as a result of the interaction between a surface mode and an internal mode which appears at the jumps of vorticity. For continuous smooth models the only analytical study we are aware of is Wright & Henyey (1989), where the instability was considered for a weakly perturbed linear profile. The physical essence of the mechanism of this wave–current interaction providing instability is identical to that of Miles (1957, 1959), which was originally developed to explain the phenomenon of wave generation by wind. A numerical study of this instability was carried out recently by Morland, Saffman & Yuen (1991)† for a few model profiles. In this section we give the formulae for the growth rates for an arbitrary (within the validity of the approach) current profile, which, contrary to the case considered by Miles (1957, 1959), are explicit. A consideration of the implications of this instability for sea wave dynamics requires more detailed information on the upper-boundary-layer fine structure than is available now and goes beyond the scope of this work.

Section 6 gives a brief discussion of the results and outlines some of their straightforward implications.

In the Appendix the problem of prime importance for the justification of the application of the results to real situations is treated – that caused by existence of very sharp boundary layers in the immediate vicinity of the surface. The problem lies in the fact that these thin layers affect noticeably the eigenfunction structure in the surface neighbourhood and cause the appearance of new large terms in the upper boundary condition. The asymptotic analysis carried out in the Appendix demonstrates that if the sharp gradients of vorticity are localized in a thin layer of characteristic scale Δ, then these large terms cancel each other and the contribution of this layer is $O(k\Delta)$. That justifies *a priori* neglect of this contribution if $k\Delta$ is small in comparison to other small parameters relevant to this problem.

2. The problem statement

2.1. *The basic equations*

We consider wave motion of ideal fluid of unit density with free surface under the influence of gravity and surface tension. Waves are assumed to be of small amplitude and imposed on the steady shear flow U uniform in the horizontal direction, $U\{U(z), V(z), 0\}$. The Cartesian frame (x, y, z) is chosen to have its origin at the unperturbed free surface, with the z-axis oriented vertically upward. We start with the Euler equations for perturbations of velocity $\boldsymbol{u}\{u, v, w\}$ and pressure p linearized upon the basic flow

$$\left.\begin{aligned} \mathrm{D}_t u + U'w + \partial_x p &= 0, \\ \mathrm{D}_t v + V'w + \partial_y p &= 0, \\ \mathrm{D}_t w + \partial_z p &= 0, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}\right\} \tag{2.1}$$

with standard boundary conditions at the free surface $z = \eta(x, y, t)$ transformed on the plane $z = 0$,

$$\mathrm{D}_t \eta = w; \quad P = (\alpha \nabla^2 + g)\eta, \tag{2.2}$$

and at the bottom $z = -H$

$$w = 0. \tag{2.3}$$

Here $\quad \mathrm{D}_t \equiv \partial_t + \boldsymbol{U}' \cdot \nabla, \quad U' \equiv \partial_z U, \quad \nabla^2 \equiv \partial_{xx} + \partial_{yy};$

and g and α are the gravity and surface tension constants, respectively.

We shall seek here solutions to (2.1)–(2.3) in the form

$$g \sim F(z)\,\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t), \tag{2.4}$$

† The author is indebted to a referee for drawing his attention to this work, which was published after the present work was submitted.

where $\boldsymbol{k}(k_x, k_y)$ is the wave vector of a chosen Fourier component, ω is its frequency in the laboratory frame of reference, $F(z)$ is a vertical distribution of the Fourier component of a certain physical value f. After eliminating pressure and horizontal velocity Fourier amplitudes we arrive at the Rayleigh equation for vertical velocity amplitude W, sometimes referred as the inviscid Orr–Sommerfeld equation

$$(\omega-\boldsymbol{U}\cdot\boldsymbol{k})(W''-k^2W)+(\boldsymbol{U}''\cdot\boldsymbol{k})\,W=0, \tag{2.5}$$

with the boundary conditions

$$(\omega-\boldsymbol{U}\cdot\boldsymbol{k})^2\,W'+[(\boldsymbol{U}'\cdot\boldsymbol{k})(\omega-\boldsymbol{U}\cdot\boldsymbol{k})-gk^2-\alpha k^4]\,W=0|_{z=0}, \tag{2.6}$$

$$W=0|_{z=-H}. \tag{2.7}$$

We note that in natural basins the direction of the basic current rotates with depth due to Earth's rotation although for comparatively short gravity waves this effect is negligible. But even in the case of parallel flow the Squire transformation, which reduces the three-dimensional problem to the two dimensional one (see Drazin & Reid 1981, p. 129), is modified in our case, as is easy to see, by the requirement of rescaling both the gravity and capillary constants:

$$g^{2D}=g(k/k_x)^2;\quad \alpha^{2D}=\alpha(k/k_x)^2\quad (k\equiv|\boldsymbol{k}|).$$

This makes it more difficult to extract a solution of the three-dimensional problem from two-dimensional numerical solutions than in case of the homogeneous problem and provides a strong additional argument in favour of developing analytical ways rather than numerical ones. It should be noted that the most important implication of the Squire theorem in the context of stability problems – the corollary that linear two-dimensional perturbations grow faster than oblique ones – holds in our case as well (the proof is given in Morland *et al.* 1991).

For simplicity, in this work we confine ourselves to consideration of the deep-water case, i.e. we put H equal to minus infinity. The study of the general case with finite H, although being interesting from the point of view of different applications, is principally the same, but the calculations are much more tedious. The results for finite-depth fluid will be reported elsewhere.

In our work we shall concentrate on the derivation of analytical solutions to the boundary-value problem (2.5)–(2.7) for a discrete spectrum mode relevant to water waves in typical oceanic conditions. We assume that there are no inflexion points in the current profile and thus the only discrete modes are the ones due to the free surface (surface waves).

2.2. *The scaling*

The problem contains a number of natural characteristic scales, which produce a set of non-dimensional parameters. Their interrelations require a special analysis.

We use wavenumber k^0 to introduce non-dimensional spatial variables $\tilde{\boldsymbol{X}}$ but keep everywhere k instead of unity for convenience. Another important spatial scale is prescribed by the characteristics scale of the basic flow vertical variability h. We note that the choice of h is not trivial and will be discussed below. Their ratio is characterized by a non-dimensional parameter s

$$s=k^0h. \tag{2.8}$$

We take as the characteristic velocity scale the value U_0 of the basic current U at the free surface. The already selected k^0 provides another velocity scale, that of wave phase velocity $c_0(k^0)$ in still water. The ratio U_0/c_0 gives a new important non-dimensional parameter γ:

$$\gamma=U_0/c_0. \tag{2.9}$$

It is convenient to present $c_0(k^0)$ in the factorized form

$$c_0 = c_{0_{gr}} Q,$$

where $c_{0_{gr}} = (g/k^0)^{\frac{1}{2}}$ is the gravity wave velocity and Q is a factor which describes the surface tension influence

$$Q^2 = 1+(k^0 B)^2.$$

Here B is the so-called Bond scale defined as $(\alpha/g)^{\frac{1}{2}}$, which gives a scale where both gravity and surface tension are of the same order. In terms of k^0 and the basic flow Froude number Fr

$$\gamma = Fr\, s^{\frac{1}{2}} Q^{-1}, \quad Fr = U_0/(gh)^{\frac{1}{2}}.$$

We note that for natural flows the Froude numbers are typically much less than unity and on rare occasions attain the order of unity.

In terms of dimensionless variables the boundary-value problem (2.5)–(2.7) takes the form

$$(C-\gamma\mathscr{U})(W''-k^2 W)+\epsilon\mathscr{U}''W = 0, \tag{2.10}$$

$$(C-\gamma\mathscr{U})^2 W' + [\mu\mathscr{U}'(C-\gamma\mathscr{U}) - C_0^2(k)\,k]\,W = 0|_{z=0}, \tag{2.11}$$

$$W = 0|_{z=-H}, \tag{2.12}$$

where $C = c/c_0$ is the dimensionless phase velocity, $Z = k^0 z$ is the dimensionless vertical coordinate, $\mathscr{U}(s^{-1}Z) = \boldsymbol{U}\cdot\boldsymbol{k}/kU_0$ is the mean flow component parallel to the direction of the chosen wave vector $\boldsymbol{k}$, $C_0^2(k) = (g/k+\alpha k)/c_0$. The notations $\boldsymbol{k}$ and k are preserved for both the dimensional and non-dimensional wave vector and its modulo.

Now the relative magnitudes of the terms are controlled by three non-dimensional parameters: γ, μ and ϵ. The form of the Rayleigh equation (2.10) contains, besides γ defined above, only the non-dimensional parameter ϵ which gives the relative magnitude of the last term and has the meaning of an integral value of the basic-flow vorticity gradient. We note that there is an essential implicit assumption in this scaling: it is supposed that the vertical scale of the solutions in question is $O(k^{-1})$ while the vertical scale of the basic flow is h. Then ϵ is formally defined as follows:

$$\epsilon = \gamma s^{-2} = Fr\, s^{-\frac{3}{2}}, \tag{2.13}$$

as it is easy to notice that in terms of original dimensional variables it is a characteristic (averaged) value of

$$U''/\omega k,$$

i.e. has the meaning of the ratio of the mean flow vorticity gradient and the product of the wave vector and wave frequency.

The boundary condition (2.11) contains, besides γ defined by (2.9), the only controlling parameter μ

$$\mu = \gamma s^{-1} = Fr\, Q^{-1} s^{-\frac{1}{2}}; \tag{2.14}$$

μ has the meaning of the ratio of the flow velocity characteristic change over the span k^{-1} to characteristic phase velocity c_0, or the ratio of the mean flow vorticity to wave frequency.

3. The exact solutions

3.1. *The eigenfunctions*

Let us seek the solution to the Rayleigh equation (2.10) with the boundary condition (2.12) in the form of an expansion in powers of ϵ

$$W(Z) = W_{(0)} + \epsilon W_{(1)} + \epsilon^2 W_{(2)} + \dots, \tag{3.1}$$

ignoring for a while the boundary condition (2.11) at the free surface. Then equating terms of the same order in ϵ we obtain the sequence of equations

$$W_{(n)}'' - k^2 W_{(n)} = \frac{\mathcal{U}''}{\gamma\mathcal{U} - C} W_{(n-1)}, \quad n = 1, \infty \tag{3.2a}$$

with the boundary condition of decay at minus infinity

$$W_{(n)} = 0|_{z=-\infty}. \tag{3.2b}$$

It should be noted that the eigenvalue C remains unspecified, while at the moment we are interested in deriving the dependence of $W(Z, C)$, i.e. with an arbitrary complex constant C.

It is convenient to present general solutions to (3.2) via the Green function G,

$$G(Z, Z') = -(1/2k)\begin{cases}\exp k(Z - Z'), & -\infty \leqslant Z \leqslant Z' \leqslant 0 \\ \exp k(Z' - Z), & -\infty \leqslant Z' \leqslant Z \leqslant 0,\end{cases} \tag{3.3}$$

in the form

$$W_{(n)}(Z) = \hat{G} W_{(n-1)}, \quad n = 1, \infty \tag{3.4}$$

where the operator $\hat{G}$ is defined as

$$\begin{aligned} \hat{G}\psi &\equiv -\frac{1}{2k}\int_{-\infty}^{0} \frac{\mathcal{U}''\psi\, \mathrm{e}^{-k|Z-z|}}{\gamma\mathcal{U} - C}\mathrm{d}z \\ &\equiv -\frac{\mathrm{e}^{-kZ}}{2k}\int_{-\infty}^{z} \frac{\mathcal{U}''\psi\, \mathrm{e}^{kz}}{\gamma\mathcal{U} - C}\mathrm{d}z - \frac{\mathrm{e}^{kZ}}{2k}\int_{z}^{0} \frac{\mathcal{U}''\psi\, \mathrm{e}^{-kz}}{\gamma\mathcal{U} - C}\mathrm{d}z. \end{aligned} \tag{3.5}$$

Then specifying normalization for the zero-order solution we obtain for the nth term of the expansion

$$W_{(n)}(Z) = \hat{G}^n \mathrm{e}^{kZ}, \quad n = 1, \infty. \tag{3.6}$$

Thus we have derived explicit expressions for all terms of the expansion. This allows one to sum formally the expansion to present the results of summation as follows:

$$\boxed{W(Z) = \frac{1}{1 - \epsilon\hat{G}}\mathrm{e}^{kZ}.} \tag{3.7}$$

Let us investigate the properties of the expansions derived for eigenfunctions with an unspecified complex constant C.

3.2. *Convergency*

First we investigate convergency of the expansion (3.1)–(3.7). We note that, using the explicit expression for the Green function (3.5), the Rayleigh equation (2.10) with the boundary condition (2.12) can be presented in a straightforward manner in the form of the Fredholm equation

$$W(Z) = -\frac{1}{2k}\int_{-\infty}^{0} \frac{\mathcal{U}''(z)\, W(z)\, \mathrm{e}^{-k|Z-z|}}{\gamma\mathcal{U}(z) - C}\mathrm{d}z \equiv \hat{G}W. \tag{3.8}$$

A solution to this equation can be found in a standard way in the form of the Neumann series under certain conditions specified below. One can easily see that our expansion (3.1)–(3.6) is the Neumann series for (3.8). It is known (e.g. Korn & Korn 1961

§15.3–8; for more details, see e.g. Tricomi 1970) that mean convergency of the Neumann series to a solution of (3.8) occurs if the kernel is bounded in the sense

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\left|\frac{\mathcal{U}''\,\mathrm{e}^{-k|Z-z|}}{\gamma\mathcal{U}-C}\right|^{2}\mathrm{d}z\,\mathrm{d}Z \equiv P < \infty, \tag{3.9}$$

and the condition

$$(\epsilon P^{\frac{1}{2}}/2k) < 1 \tag{3.10}$$

is fulfilled. If, besides that, the kernel satisfies the conditions

$$\int_{-\infty}^{0}\left|\frac{\mathcal{U}''\,\mathrm{e}^{-k|Z-z|}}{\gamma\mathcal{U}-C}\right|^{2}\mathrm{d}Z < \infty, \quad \int_{-\infty}^{0}\left|\frac{\mathcal{U}''\,\mathrm{e}^{-k|Z-z|}}{\gamma\mathcal{U}-C}\right|^{2}\mathrm{d}z < \infty, \tag{3.11}$$

the Neumann series converges uniformly in $[-\infty, 0)$. The conditions (3.9)–(3.11) are clearly satisfied when ϵ is small and there are no poles in the integrand lying on the real axis or in certain neighbourhood of the real axis ($\mathrm{e}^{-1/\epsilon}$, the neighbourhood for the first-order poles). But the solutions with values of C lying in a certain range of ϵ near the real axes with real part C_{R} lying within the flow velocity range $[\mathcal{U}_{\min}, \mathcal{U}_{\max}]$ are of great interest for us. As the convergency criteria (3.9)–(3.11) are only sufficient ones and are too restrictive in the context of our problem, let us try to prove convergency of our series for this range of C after lifting the most essential restrictions.

Starting with (3.4) in the explicit form

$$W_{(n)}(Z) = \frac{1}{2k}\int_{-\infty}^{0}\frac{\mathcal{U}''\,W_{(n-1)}\,\mathrm{e}^{-k|Z-z|}}{\gamma\mathcal{U}-C}\mathrm{d}z$$

one can obtain in a straightforward manner an inequality relating the maximal values of $|W_{(n)}(Z)|$ denoted as $m_{(n)}$:

$$m_{(n)} \leqslant m_{(n-1)}\frac{1}{2k}\max_{Z}\left|\int_{-\infty}^{0}\frac{\mathcal{U}''\,\mathrm{e}^{-k|Z-z|}}{\gamma\mathcal{U}-C}\mathrm{d}z\right| \equiv \frac{m_{(n-1)}I}{2k}, \tag{3.12}$$

where $\max_Z$ means maximal-over-Z value of the integral. The series $\epsilon^{n-1}m_{(n-1)}\ldots\epsilon^{n}m_{(n)}\ldots\epsilon^{n+1}m_{(n+1)}$ majorates the functional expansion for $W_{(n)}(Z)$ and is a geometric progression. Its nth term can be easily found (with ϵ^n normalization taken into account) to be

$$\epsilon^{n}m_{(n)} = (\epsilon I/2k)^{n}. \tag{3.13}$$

The convergence of this series is provided by the obvious condition

$$\epsilon_{\mathrm{true}} = (\epsilon I/2k) < 1. \tag{3.14a}$$

The condition (3.14*a*) gives us the true key small parameter of the problem, we designate it ϵ_{true}, and it does not depend on the way we non-dimensionalized the variables. In terms of the original dimensional variables our convergency criterion takes the form

$$\boxed{\epsilon_{\mathrm{true}} = \max_{Z}\left|\frac{1}{2k}\int_{-\infty}^{0}\frac{U''(z)\,\mathrm{e}^{-k|Z-z|}}{U(z)-c}\mathrm{d}z\right| < 1.} \tag{3.14b}$$

We stress the integral character of both criteria (3.10)–(3.11) and (3.14*a*). That means that for convergency the relative magnitude of the last term in the Rayleigh equation should not necessarily be small or even finite everywhere in the fluid, smallness in the integral sense only is required.

We also mention that (3.13) under condition (3.14*a*) gives the bound for the residual term of the truncated expansion (3.1)–(3.6) as well.

Evidently the condition (3.14*a*) is much milder than (3.10)–(3.11), however our aim now is not to find the maximal convergency radius from (3.14*a*) (a detailed analysis of this inequality lies beyond the scope of this work) but just to prove this fact for the range of interest. It is sufficient for these purposes to strengthen, and thus to simplify, the inequality (3.12), inserting I_0 for I into (3.12) and its consequent equations (3.13), (3.14*a*), where

$$I_0 \equiv \left| \int_{-\infty}^{0} \frac{\mathscr{U}''}{\gamma\mathscr{U}-C}\,\mathrm{d}z \right|. \tag{3.15a}$$

Let us consider in detail the case $\operatorname{Re} C$ lying within the flow velocity range $\operatorname{Re} C \subset [\mathscr{U}_{\min}, \mathscr{U}_{\max}]$. Then I_0 can be evaluated as follows:

$$I_0 = (A^2+B^2)^{\frac{1}{2}},$$

where
$$A \equiv \mathrm{PV}\int_{-\infty}^{0} \frac{\mathscr{U}''}{\gamma\mathscr{U}-\operatorname{Re} C}\,\mathrm{d}z, \quad B = \frac{\pi \mathscr{U}''_c}{\mu \mathscr{U}'_c} \tag{3.15b}$$

and subscript 'c' means that derivatives of $\mathscr{U}$ are taken at the point of the pole of the integrand in (3.15). The function $\mathscr{U}$ is assumed to be analytically continued. As γ is of the order of unity in this range one can expect A also to remain of the order of unity. The contribution due to the complex part, B, is $O(\mu^{-1})$ and is clearly the main term in the case of small μ. Then $I_0 \approx B$ and using this estimate for ϵ_{true} we get a simplified form of the criterion (3.14*a*) in terms of non-dimensional variables: $s > \frac{1}{2}\pi$, or in dimensional variables

$$\epsilon_{\text{true}}^{\text{est}} = \frac{\pi U''_c}{U'_c 2k} < 1. \tag{3.16}$$

When s is large enough condition (3.16) is fulfilled; then we can conclude that convergency holds for arbitrary complex C, however small its imaginary part. Thus we have shown that the expansion (3.1)–(3.6) is absolutely and uniformly convergent provided ϵ_{true} is small.

Summarizing this subsection we conclude: the smallness of ϵ_{true} in the sense (3.14*a*) provides absolute and uniform in $z \subset [0, -\infty]$ convergence of the series (3.6) for an arbitrary C. The condition of convergency (3.14*a*) is much milder than the general conditions (3.10)–(3.11). In fact, ϵ_{true} can differ considerably from ϵ in the spectral range where $\operatorname{Re} C$ lies in the interval $[U_{\min}, U_{\max}]$ on the real axes and very close to $(O(\mathrm{e}^{-1/\epsilon}))$ the real axes. In the latter intervals we cannot prove the convergence. For convenience of further analysis we can choose the characteristic scale h so that for the k-values under consideration $\epsilon_{\text{true}} = \epsilon$.

3.3. *The eigenvalues*

We have found the exact solution to the Rayleigh equation with zero boundary condition at infinity in the form of the series (3.1)–(3.7) and proved its convergence. We note that this solution is of interest in itself and can be used in another contexts as well. To find the eigenvalue and thus to specify the uncertain complex constant in (3.6), the free surface boundary condition (2.11) should be satisfied. We write it as an implicit function of C and k:

$$\mathscr{L}(W(0), W'(0), C, k, g, \alpha) = 0, \tag{3.17}$$

where the dependence of $\mathscr{L}$ on the parameters g and α is explicit, and $W(0)$, $W'(0)$ are assumed to be known functions of C given by (3.6). We note that $W(z, C)$, $W'(z, C)$ are analytic functions on the C-plane, except for the cut $[U_{\min}, U_{\max}]$ on the real axes and in the $O(e^{-1/\epsilon})$ vicinity of the extremal current values on the real axes, where the convergence is not proved. The remarkable property of $W(Z)$ is

$$\boxed{\begin{aligned} &W'_{(n)}(Z)|_{Z=0} = -kW_{(n)}(Z)|_{Z=0}, \quad n \geqslant 1; \\ &W'_{(0)}(Z) \equiv k e^{kZ} = kW_{(0)}(Z), \quad n = 0; \end{aligned}} \tag{3.18}$$

which allows one to introduce a single function $R(C)$ instead of $W(0)$ and $W'(0)$ as follows

$$\boxed{W(0) = 1+\epsilon R; \quad W'(0) = k(1-\epsilon R),} \tag{3.19}$$

where

$$R = \sum_{n=1} \epsilon^{n-1}(-2k)^{-n} \int_{-\infty}^{0} \mathrm{d}z\, e^{kz} f \int_{-\infty}^{0} \mathrm{d}z_1\, e^{-k|z-z_1|} f \int_{-\infty}^{0} \dots \int_{-\infty}^{0} \mathrm{d}z_n\, e^{-k|z_{n-1}-z_n|} f \tag{3.20}$$

and

$$f \equiv \mathscr{U}''/(\gamma\mathscr{U} - C). \tag{3.20}$$

This enables us to present the final equation for C in a much more convenient form:

$$\boxed{(C-\gamma\mathscr{U}(0))^2 k(1-\epsilon R) + [\mu\mathscr{U}'(C-\gamma\mathscr{U}(0)) - C_0^2(k)k](1+\epsilon R) = 0.} \tag{3.21}$$

As the function R can be straightforwardly calculated numerically for any given profile as a function of k and C, the solution of (3.21) can be easily found numerically. We note that, on employing (3.19), all the boundary conditions of type (1.3) can be easily presented in the form

$$R(k, C) = \Phi(C, k),$$

where $\Phi(C, k)$ is an *a priori* known function straightforwardly expressed in terms of $F_1(k, C), F_2(k, C)$. Thus we have got at least a new perspective on the numerical treatment of the problem. Evidently however, for many purposes it is more preferable to deal with an analytical solution.

It is natural to seek an analytical solution of (3.17) or (3.21) in the form of a power series in ϵ:

$$C(k, g, \alpha) = C_{(0)}(k, g, \alpha), + \epsilon C_{(1)}(k, g, \alpha) + \dots. \tag{3.22}$$

To justify this approach one should prove that $\mathscr{L}$ is an infinitely differentiable function of C, ϵ and k. To prove this it is enough to check the differentiability of R with respect to C, ϵ and k. It is easy to see from (3.20) that $R(C)$ is an analytic function unless there is a pole of f in the integrand on the real axes. Assuming that there are no such poles and inserting (3.22) in (3.21), we obtain an explicit expression for the nth term of series (3.22):

$$C_{(n)} = \frac{2k\left[-\sigma_{(n)} + \sum_{m=2}^{n-1} \rho_{(n-m-1)}\sigma_{(m)}\right] + \mu\mathscr{U}' \sum_{j=1}^{n-1} \rho_{(n-j-1)} C_{(j)} + [C_S \mu\mathscr{U}' - C_0^2]\rho_{(n-1)}}{2kC_S + \mu\mathscr{U}'}, \tag{3.23a}$$

where $\sigma_{(m)}$ is the term of mth order in ϵ in the sum

$$\sigma_{(m)} \equiv \sum_{j=1}^{m-1} C_{(m-j)} C_{(j)}; \tag{3.23b}$$

$\rho_{(j)}$ is the term of jth order in ϵ of the function R given by (3.20),

$$\rho_{(j)} = \sum_{i=1}^{j-1} R_{(i)}(C_{(0)})\, C_S^{i-j-1} \sum C_{(1)}^{\alpha 1} \dots C_{(m)}^{\alpha m}\, \delta(\textstyle\sum p\alpha_p - j + i - 1); \qquad (3.23\,c)$$

$C_{(0)}$ is the zero-order solution given by

$$(C_{(0)} - \gamma \mathscr{U}_0) = -\mu \mathscr{U}_0'/2k \mp [(\mu \mathscr{U}_0'/2k)^2 + C_0^2(k)]^{\frac{1}{2}}; \qquad (3.24)$$

and C_S denotes $(C_{(0)} - \gamma \mathscr{U}_0)$.

The zeroth-order solution (3.24) has two real branches for arbitrary k. The + and − branches correspond to waves propagating along and opposite to the current, respectively, in the frame of reference moving with the maximal flow velocity. The same two-branch structure of the solution is preserved at all orders in ϵ, the only difference being that the higher-order corrections $C_{(n)}$ to these two branches can be complex.

Thus, having found explicit formulae for $C_{(n)}$ and with the convergence of the series (3.22) being guaranteed by the conditions mentioned above, we have solved the problem: the converging series (3.6), (3.22) taken as a whole provide exact solutions for eigenfunctions and eigenvalues, while the truncated sums enable one to find the approximate solutions with *a priori* prescribed accuracy.

Let us consider the range of validity in k-space of the solutions derived. We recall that the procedure is based on the assumptions of convergence of the series (3.7) for W or R and its infinite differentiability, which in its turn is provided by the assumptions of the smallness of ϵ and the absence of poles lying on the real axes in the vicinity of the k we are interested at the moment. Thus we have no problems with convergency and differentiability of W. The only questionable range is the $O(\mathrm{e}^{-1/\epsilon})$ outer vicinity (in terms of C) of the ends of the current velocity interval. This range requires special consideration, which goes beyond the scope of this work. We expect a break in continuity of $C(k)$ at these points.

3.4. *The asymptotics for some spectral ranges*

In the preceding subsection we have derived exact solutions to the boundary-value problem (2.10)–(2.12) in the form of rapidly converging power series in ϵ (3.6), (3.23a), where the orders of the parameters μ and γ in terms of ϵ have not been specified and were implicitly assumed to be of the order of unity. However for the wide range of wave scales in typical ocean conditions at least one of these parameters is small. Specifying this smallness in terms of ϵ for a particular range of wave scales one can obtain another series in ϵ for the phase velocity C. One can most easily specify the relations among the parameters μ, γ and ϵ in two ranges of k, whch we call for definiteness the 'medium-wave range' and the 'short-wave range', and shall define these below.

First we recall that

$$\mu/\gamma = s^{-1}, \quad \mu/\epsilon = s, \quad \mu^2/\gamma = \epsilon, \qquad (3.25)$$

where $s \equiv kh$.

Medium-wave range: we shall use this term for the wave band characterized by $s = O(1)$. Hence for this range

$$\mu \approx \gamma \approx \epsilon. \qquad (3.26)$$

We define *short-wave-range* by the inequality $s \gg 1$. In this wave band $\gamma \leqslant O(1)$, the large parameter s makes both ϵ $(= \gamma s^{-2})$ and μ $(= \gamma s^{-1})$ small. Thus in terms of ϵ:

$$\gamma \approx O(1), \quad \mu \approx O(\epsilon^{\frac{1}{2}}). \qquad (3.27)$$

One can also specify other wave bands with different relations among the small parameters, which might be of interest for geophysical applications, but the two selected by us seem to be of most importance and in the next section we shall confine our analysis mainly to their consideration. We do not present the expressions for the corresponding series here, as they prove to be more bulky than the original ones (3.23). The advantage of using these relations among the small parameters is revealed only when dealing with the truncated sums of low order.

4. Approximate solutions: the leading-order terms

In the preceding section we have derived the explicit formulae (3.7), (3.23*a*) for the exact solutions to the boundary-value problem (2.10)–(2.12) in the form of rapidly converging power series in ϵ. For many practical purposes it is sufficient to confine ourself to consideration of several lowest-order terms only. This section contains a brief analysis of the approximate solutions to the boundary-value problem (2.10)–(2.12) given by the terms of leading-order in ϵ for different ranges of μ and γ.

4.1. *The zeroth-order approximation*

Let us start with the zeroth-order solutions:

$$W_{(0)} = \mathrm{e}^{kZ}, \tag{4.1}$$

$$C_{(0)}(k) = \gamma\mathcal{U}_0 - \mu\mathcal{U}_0'/2k \mp [(\mu\mathcal{U}_0'/2k)^2 + C_0(k)^2]^{\frac{1}{2}}. \tag{4.2a}$$

At this order of approximation the eigenfunction is given by the potential theory and does not depend on the presence of the shear flow at all, while the expression for eigenvalue 'feels' the shear flow only through its two characteristics at the surface, namely $\mathcal{U}_0$ and $\mathcal{U}_0'$, and is the same as the dispersion relation for a constant-vorticity flow (Biesel 1950; Peregrine 1976; Jonsson 1989). If μ remains small in the range of interest, (4.2*a*) degenerates into the standard still-water dispersion relation, corrected by taking account of the Doppler shift for the short-wave range. Thus at this order the presence of the current reveals itself mainly in a Doppler shift for short waves. The shear gives a contribution of approximately μ, i.e. $O(\epsilon^{\frac{1}{2}})$ in the short-wave range:

$$C_{(0)}(k) = \gamma\mathcal{U}_0 - \mu\mathcal{U}_0'/2k \mp C_0(k). \tag{4.2b}$$

Sometimes it is more convenient to attribute the $O(\epsilon^{\frac{1}{2}})$ terms to the next-order approximation.

4.2. *The first-order approximation*

For the first-order (in ϵ) solutions we have from (3.6), (3.23*a*), taking (4.2) into account:

$$W_{(1)} = (1+\epsilon\hat{G})\,\mathrm{e}^{kZ}, \tag{4.3}$$

$$C_{(1)} = \frac{[(C_{(0)}-\gamma\mathcal{U}_0)\,\mu\mathcal{U}' - C_0^2]\,\rho_{(0)}}{2k(C_{(0)}-\gamma\mathcal{U}_0)+\mu\mathcal{U}'}$$

$$= \frac{[(-\mu\mathcal{U}_0'/2k \mp [(\mu\mathcal{U}_0'/2k)^2 + C_0(k)^2]^{\frac{1}{2}})\,\mu\mathcal{U}' - C_0^2]\,\rho_{(0)}}{\mp 2k[(\mu\mathcal{U}_0'/2k)^2 + C_0(k)^2]^{\frac{1}{2}}}. \tag{4.4a}$$

It is easy to see that the first-order phase velocity correction (4.4*a*) both in the medium- and short-wave ranges reduces to

$$C_{(1)} = \frac{-C_0\rho_{(0)}}{2k} + O(\epsilon). \tag{4.4b}$$

We recall that in the general case

$$\rho_{(0)} = \int_{-\infty}^{0} dz \frac{\mathscr{U}''}{\gamma\mathscr{U} - C_{(0)}} e^{2kz} \tag{4.5}$$

and the difference between the 'medium' and 'short' waves lies in the specific asymptotic for $C_{(0)}$ and the way of calculating the integral. One should also bear in mind that the next-order terms of the $C_{(0)}$ expansion in ϵ which contribute to C in this order are different for different ranges.

For medium-range waves it can be taken with the desired accuracy as

$$-C_0 \rho_{(0)}^{\mathrm{mw}} \approx \int_{-\infty}^{0} dz\, \mathscr{U}'' e^{2kz} + O(\epsilon) \approx (4k^2) \int_{-\infty}^{0} dz\, \mathscr{U} e^{2kz} + O(\epsilon). \tag{4.6}$$

The latter formula (combined with (4.4*b*)) is the same as the result of Stewart & Joy (1974).

For the short waves the most important feature of the expression (4.5) for $\rho_{(0)}$ is that the integrand generally contains poles where $C_{(0)}$ equals $\mathscr{U}(z)$. The path of integration, which formally goes strictly along the real axes, should be chosen as is common for these types of hydrodynamic problems, treating the zeroth-order eigenvalue C as the real limit of the corresponding Laplace index in the Cauchy problem (e.g. Dikii 1976; Lin 1966; Drazin & Reid 1981), i.e. the eigenmodes are treated as the $t \Rightarrow \infty$ limits of the Cauchy problem. That means that taking the integral we assume

$$\operatorname{Im} C \operatorname{sgn} k > 0,$$

Putting $\operatorname{Im} C$ to zero upon integration, or alternatively

$$\frac{1}{\gamma\mathscr{U} - C} = \mathrm{PV}\frac{1}{\gamma\mathscr{U} - C} + \mathrm{i}\pi\delta(\gamma\mathscr{U} - C) = \mathrm{PV}\frac{1}{\gamma\mathscr{U} - C} + \frac{\mathrm{i}\pi\delta(Z - Z_c)\operatorname{sgn} k}{\gamma|\mathscr{U}'|}. \tag{4.7}$$

Thus
$$\rho_{(0)}^{\mathrm{sw}} = \mathrm{PV}\int_{-\infty}^{0} dz \frac{\mathscr{U}''}{\gamma\mathscr{U} - C_{(0)}} e^{2kz} + \mathrm{i}\frac{\pi\mathscr{U}''(Z_c)\, e^{2kZ_c} \operatorname{sgn} k}{\mu|\mathscr{U}'(Z_c)|}, \tag{4.8}$$

where Z_c is a critical point in the sense $\gamma\mathscr{U} = C_{(0)}$. If there is more than one critical point, then the summation over all the critical points in the second term in (4.8) is implicitly assumed.

We note that from the formal point of view our boundary-value problem is complex self-conjugated and therefore necessarily possesses two complex-conjugated solutions; however we, by specifying the integration path, have chosen from the two complex-conjugate solutions the one dictated by external (physical) reasoning. We shall not dwell upon the details of the justification of this choice, which can be presented in variety of ways (see e.g. the discussion of a similar problem in Lin 1955, §8; Dikii 1976; Craik 1985). We select by that way only the growing modes, while their decaying complex-conjugates counterparts have no clear physical value as they are not the limit of the Cauchy problem and also cannot be derived as a uniform non viscous limit of the viscous problem.

We recall that the imaginary part of the eigenvalues. like their real part, is given by the converging series (3.23*a*) but starting with first instead of zero order in ϵ. To understand the physical implementations of complex eigenvalues it will be sufficient to investigate the leading-order terms in ϵ.

We also recall that we are working in a specifically chosen coordinate frame, where the $\boldsymbol{x}$-axis is directed along $\boldsymbol{k}$ and thus $\operatorname{sgn} k$ is always positive in this frame. We stress another implication of this choice: for a critical level to occur $\mathscr{U}$ should be negative.

The integral in (4.8) can be also simplified by utilizing the presence of the large parameter s (we recall that $\mathscr{U}$ is function of $s^{-1}Z$):

$$\mathrm{PV}\int_{-\infty}^{0} \mathrm{d}z \frac{\mathscr{U}''}{\gamma\mathscr{U}-C_{(0)}}\mathrm{e}^{2kz} = \frac{1}{2k}\frac{\mathscr{U}''(0)}{\gamma\mathscr{U}(0)-C_{(0)}}+O(s^{-\frac{1}{2}}). \tag{4.9}$$

Finally the expression for C in the short-wave range is

$$C = \pm C_0(k) - \gamma\mathscr{U}_0 + \frac{-\mu\mathscr{U}_0'}{2k} - \epsilon\frac{C_0\rho_{(0)}}{2k} \mp \frac{1}{2C_0}\left(\frac{\mu\mathscr{U}_0'}{2k}\right)^2 \tag{4.10}$$

$$O(\epsilon^0) \qquad\qquad O(\epsilon^{\frac{1}{2}}) \qquad\qquad O(\epsilon^1)$$

where $\rho_{(0)}$ is given by (4.8), (4.9). It should be noted that for 'short waves' the real part of C is expressed (apart from the still-water velocity C_0) in terms of the flow velocity and its first and second derivatives at the surface only, while the imaginary part is determined by the first and second derivatives of the flow velocity at the critical levels.

The higher-order approximate solutions can be easily deduced from (3.6), (3.23*a*) for any particular wave scale range in a similar way. The analysis of these higher-order approximations goes beyond the aims of this work.† Here we focus our attention on some physical consequences of these leading-order approximate solutions.

5. The instability

It is easy to see from the solutions derived above that the eigenvalues become complex due to the presence of the critical layers.

We recall that the imaginary part of the eigenvalues, analogously to their real part, is given by the converging series (3.23*a*) but starting with first instead of zero order in ϵ. To understand the physical implementation of complex eigenvalues it will be sufficient to investigate the leading-order terms in ϵ.

The existence of the specific instability in a shear flow due to the free surface is well known, but was previously analysed mainly using the piecewise-linear models (e.g. Gertenshtein *et al.* 1988). These models correctly predict the fact of instability and can also give correct quantitative estimates of the growth rates if the number of constant-vorticity layers N is large enough. But for the range of instability in k-space, these models predict the intermittent layered structure of $N-1$ zones of instability separated by the stability islands, which is physically inadequate. The only two exceptions that do deal with the instability in the case of a smooth current profile are Morland *et al.* (1991), where three model current profiles are treated numerically; and Wright & Henyey (1989) where the instability has been described analytically by employing a perturbation technique near the linear velocity profile. In this section we focus our attention on the basic features of this instability, which follow immediately from the lowest-order solutions.

The questions we shall try to clarify in this section are: When does a real physical instability occur? What is the physical nature of this instability and what are its specific features?

The answer to the first question follows straightforwardly from the formulae (4.8)–(4.10):

$$\operatorname{Im} C = -\epsilon C_0 \frac{\pi\mathscr{U}''(Z_c)\,\mathrm{e}^{2kZ_c}}{\mu|\mathscr{U}'(Z_c)|\,2k}, \tag{5.1}$$

† The second-order terms for the medium-wave range have been reported recently by Kirby & Chen (1989), and for an arbitrary range in the case of arbitrary fluid depth will be given elsewhere.

i.e. we have decay, if $\mathscr{U}''(Z_c)$ is positive, and instability, in which we are more interested, if

$$\mathscr{U}''(Z_c) < 0. \tag{5.2}$$

We recall that $\mathscr{U}(Z_c)$ is the flow velocity projection on $\boldsymbol{k}$ at the level Z_c and is always negative due to our choice of the frame of reference (see the comments on (4.8)).

The physical mechanism of this wave–current interaction providing wave instability is identical to that of Miles (1957, 1959), which was originally developed to explain the phenomenon of wave generation by wind. It should be noted that in our case increments are $O(\epsilon)$, while Miles' increments are of the order of the ratio of the densities of air and water, which is much smaller. The leading-order term for the growth rate Γ can be finally presented as

$$\Gamma = \frac{\operatorname{Im} C}{C_{(0)}} = -\epsilon \frac{C_0}{C_{(0)}} \frac{\pi \mathscr{U}''(Z_c)\, e^{2kZ_c}}{\mu |\mathscr{U}'(Z_c)|\, 2k} \tag{5.3}$$

or in terms of the original dimensional variables

$$\Gamma \approx +\frac{1}{1 - U_0/C_0} \frac{\pi U''(Z_c) \cdot \boldsymbol{k}\, e^{2|\boldsymbol{k}|Z_c}}{|U'(Z_c) \cdot \boldsymbol{k}|\, 2|\boldsymbol{k}|}. \tag{5.4}$$

The necessary condition for the instability to occur is the existence of a critical layer. It is easy to see from the relation prescribing the critical-layer depth

$$C_{(0)}(k) = U_0 - [U_0'/2k + C_0(k)] = U(Z_c) \tag{5.5}$$

that this condition takes the form

$$C_0(k) \leqslant U_0 - U_0'/2k. \tag{5.6}$$

Thus the critical layer occurs for waves propagating in the direction opposite to the surface current with phase velocities smaller than a threshold value given by (5.6), which in its turn is necessarily smaller than the maximal current value. In the context of wind waves upon wind-driven drift currents this means that this instability occurs for the waves running against the wind. We stress that this mechanism of instability has nothing in common with the instabilities of shear flows with rigid-lid boundaries, where 'internal' modes are growing and the criteria of their instability are related to the presence of the inflexion points in the current profile.

From (5.5) one can also infer the important fact that the critical layer never reaches the surface, even if we put C_0 equal to zero. In reality, due to capillarity, C_0 always exceeds a certain minimal value, which for clean water is approximately 23 cm/s.

Let us evaluate the maximal increment values and specify their positions in k-space. Consider for example a model exponential current profile:

$$U = U_0\, e^{-z/h}.$$

Then (5.5) takes the form

$$C_{(0)}(k) = U_0(1 - 1/2kh) - C_0(k) \equiv U_0 \chi = U_0\, e^{-z_c/h}, \tag{5.7}$$

where $C_0(k) = (g/k + \alpha k)^{\frac{1}{2}}, \chi \equiv (1 - 1/2kh) - C_0/U_0$. This enables one to obtain the following simple explicit formula for the critical-layer depth:

$$Z_c = h \ln [(1 - 1/2kh) - C_0/U_0] = h \ln \chi, \tag{5.8}$$

and, finally

$$\Gamma = (\pi/2kh)\,(U_0/C_0(k)^{-1}\, \chi^{2kh-1}. \tag{5.9}$$

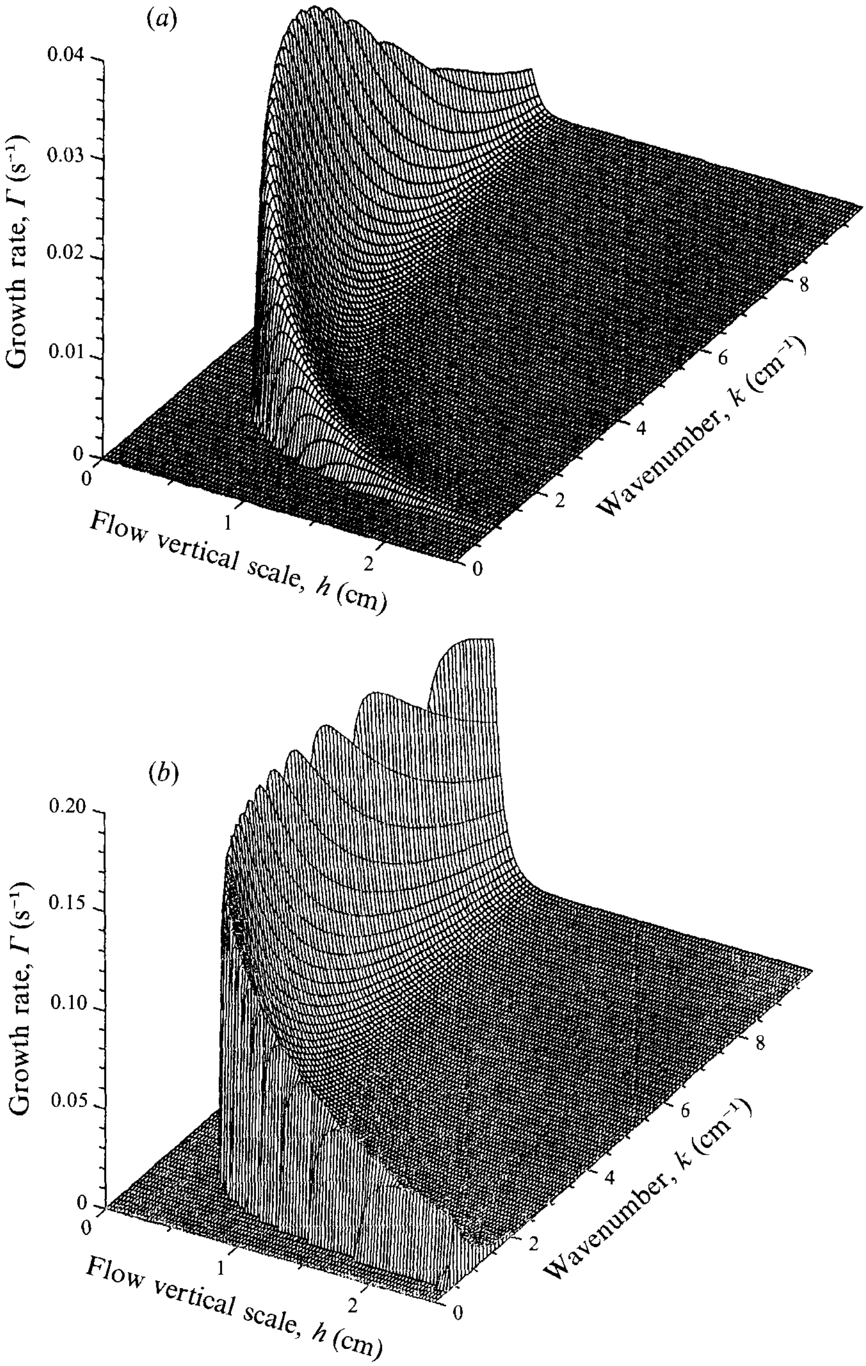

FIGURE 1. Growth rate versus wavenumber k and the flow vertical scale h for the model with an exponential current profile. (a) $U_0 = 0.5$ m/s; (b) $U_0 = 1.0$ m/s.

The formula gives very sharp maximum of $|\Gamma|$ at certain k^*, while the peak value of Γ, its actual order of magnitude, also strongly depends on the current parameters U_0, h (see figure 1). The aim of the figure is just to give a qualitative idea of this dependence and to illustrate two main points: first, that at realistic velocity values the growth rates can far exceed the Miles wind-induced growth rates;† second, the very strong sensitivity of the growth rates to the vertical structure of the flow. The latter means that, to judge the implications of this instability for sea wave dynamics, much more detailed information on the upper-boundary-layer fine structure is required than is available now. Synthesis of the often controversial data on the upper-boundary-layer fine structure and the detailed investigation of the instability within the context of wind-wave dynamics requires a special study and goes beyond the scope of this work. The only conclusion that can be made at the moment is that the instability might be important for gravity–capillary wave dynamics and in our opinion merits this special study.

6. Discussion

First we briefly summarize the main results of the paper and then outline some of their straightforward implications.

Exact solutions to the Rayleigh equation are found in the form of an absolutely converging Neumann series in powers of ϵ, which exists under the relatively mild condition (3.16). On the basis of this solution we also solve exactly the boundary-value problem in terms of the series in ϵ. The parameter ϵ characterizes the smallness of the wave motion deviation from the potential one. This deviation is commonly believed to be small for the whole range of wind-wave scales and current parameters relevant to the ocean conditions, which implies not only absolute convergence of the series, but moreover very rapid convergence.

However, because of the lack of good quality field data on the vertical structure of the current we are at present unable to check whether the wind waves are nearly potential and to conclude definitely for what range of scales the approach is really applicable. The problem lies in the fact that in the immediate vicinity of the surface there are water boundary layers with very sharp current gradients (see Csanady 1984 for field observations and e.g. Lin & Gad-el-Hak 1984 for laboratory measurements). Within this layer wave motion is essentially non-potential but the question is whether it is thin enough to be neglected. The asymptotic analysis carried out in the Appendix demonstrates that if the sharp gradients of vorticity are localized in a thin layer of, say, scale Δ then the new large terms due to this layer cancel each other and the contribution of this layer to the dispersion relation is $O(k\Delta)$. That justifies *a priori* neglect of this contribution if and only if $k\Delta$ is small in comparison to ϵ. We keep the question about the range of applicability of our results open, but want to emphasize that the question of the validity of this approach relates to water wave near-potentiality. A special study of the existing data to answer this question is now in progress.

We note that the smallness of the deviation from potentiality does not mean that the shear contribution to the dispersion relation is unimportant; shear can considerably affect group velocity and especially its derivatives. Moreover the presence of shear produces important qualitative changes in the wave dispersion, besides the appearance of the new continuous spectrum modes. The dispersion law becomes anisotropic and (in the short-wave range) complex. Simple explicit formulae for the growth rate allow one to describe the physically important phenomena of the instability of gravity–capillary waves propagating against the wind. In spite of the fact that the increments in a certain range of scales can exceed the Miles increments caused by wind,

† We note that the maximum of Γ in the (k, h)-plane lies beyond the range of validity of (5.9), but the curves give an adequate qualitative picture.

for a decisive conclusion about their role one should base the calculations upon the good quality data on the fine structure of the water boundary layer. This task will be done elsewhere.

Among the most obvious applications of the expressions obtained for $\omega(\boldsymbol{k})$ the first is connected with wave propagation in slow inhomogeneous currents in horizontal directions, where the WKB description of wave transformation is relevant. In this class of problems wave evolution is governed by the set of Hamiltonian equations with $\omega(\boldsymbol{k}, U(x,y))$ being Hamiltonian. Having an explicit expression for $\omega(\boldsymbol{k})$ allows one to treat these problems analytically, while an explicit expression for $W(z)$ makes possible an analytical calculation of an adiabatic invariant (wave action) (e.g. Voronovich 1976). The results (the presence of the complex eigenvalue range) also might give a basis for extension of the classical WKB set of equations to the complex ray equations.

A quite different, but still straightforward, possible important application is to use this perturbation-like procedure to solve the inverse problem, i.e. the problem of obtaining a shear current vertical profile $U(z)$ from the $C(\boldsymbol{k})$ dependence, assumed known.

Another area of possible application and extension of the results is connected with the Rayleigh equation or a Rayleigh-like equation with different boundary conditions which arise in other problems of hydrodynamics (waves in shear flows with elastic boundaries, barotropic Rossby waves on a jet, etc.). The most obvious example of this kind of extension is the boundary-value problem for water waves with Rayleigh viscosity, which is relevant to waves in narrow channels with the sidewall friction taken into account. However, generally each problem requires special consideration: one should check whether the deviation from potential motion is small in the range of parameters of physical interest.

This work was accomplished while the author was a guest of the Royal Netherlands Meteorological Institute (KNMI), De Bilt. The author is grateful to P. A. E. M. Janssen for helpful comments on the manuscript and to A. Voorrips for drawing the author's attention to a number of misprints in the first draft of the work. The hospitality of Institute de Mécanique Statistique de la Turbulence at Marseille (Luminy), where a revision of the present work was done is appreciated.

Appendix

To investigate the effect of thin layers with sharps gradients of vorticity we consider a current of boundary-layer type: we assume intense vorticity to be localized within a certain thin layer of characteristic width Δ,

$$k\Delta = \delta \ll 1, \tag{A 1}$$

while within the main body of the fluid vorticity gradients are small so that the parameter ϵ_{true} defined by (3.14*b*) is smaller than unity, which validates the solution to the Rayleigh equation in the form (3.1). In the boundary layer the term with U'' is large, the first and the last terms in the Rayleigh equation nearly balance each other, while the term $\sim k^2 W$ acts as a perturbation. Within the layer the solutions to the Rayleigh equation are far from near-potential ones and are given by the Heisenberg expansion in powers of δ^2 (see e.g. Drazin & Reid 1981). The matching of these asymptotics gives one a uniformly valid (in $z \subset [0, \infty]$) presentation of the solution in question. One can use here either aysmptotic technique; we shall follow the line developed in Shrira (1989).

It is convenient to present the current profile in the form

$$U(z) = U_{\text{bl}}(z/\delta)\, U^0(z/s),$$

where we choose $U_{\text{bl}}(z/\delta) \Rightarrow 1$ as $(z/\delta) \Rightarrow \infty$, while the function $U^0(z/s)$ describes the basic current with the boundary layer removed. Then one can easily check that the leading-order term in the Rayleigh equation has the form

$$W(z) = (U_{\text{bl}}(z/\delta) - C/U^0(0)\, \Phi^0(z/s, C, k) + O(\delta), \tag{A 2}$$

where Φ^0 is the solution given by (3.7) for the profile $U^0(z/s)$ satisfying the zero boundary conditions at infinity. Inserting (A 2) into the boundary condition at the free surface one can easily see that all the leading-order terms caused by the shear boundary layer, i.e. all the terms containing $U_{\text{bl}}(z/\delta)$ and its derivatives, cancel each other at zero order in δ. Thus we have again arrived at the boundary-value problem in terms of Φ^0, the solution of which has been derived in §3. Hence the dispersion relation $C(k)$ corresponds to the profile with the boundary layer neglected.

One can conclude that the presence of the boundary layer contributes a term of $O(\delta)$ into the dispersion relation. The effect can be neglected if $\delta \ll \epsilon_{\text{true}}$ and we are satisfied with the accuracy provided by the leading-order terms. When δ and ϵ_{true} are small and comparable one should calculate the first-order term in δ as well as in ϵ in order to find the correction to the still-water dispersion relation.

REFERENCES

ABDULLAH, A. J. 1949 Wave motion at the surface of a current which has an exponential distribution of vorticity. *Ann. N.Y. Acad. Sci.* **51**, 425–441.

BIESEL, F. 1950 Etude thereorique de a houle en eau courante. Houille Blanche, vol. 5, Numero Special A, pp. 279–285.

CRAIK, A. D. D. 1985 *Wave Interactions and Fluid Flows*. Cambridge University Press.

CSANADY, G. T. 1984 The free surface turbulent shear layer. *J. Phys. Oceanogr.* **14**, 402–411.

DIKII, L. A. 1976 *Hydrodynamic Stability and Atmosphere dynamics*. Leningrad: Hydrometeoizdat (in Russian).

DRAZIN, P. G. & REID, W. H. 1981 *Hydrodinamic Stability*. Cambridge University Press.

FENTON, J. D. 1973 Some results for surface gravity waves on shear flows. *J. Inst. Maths Applics* **12**, 1–20.

GERTSENSHTEIN, S. YA., ROMASHOVA, N. B. & CHERNYAVSKI, V. M. 1988 On the generation and development of wind waves. *Izv. Akad Nauk SSSR Mech. Zhid i Gaza* **3**, 163–169 (in Russian).

HENYEY, F. & WRIGHT, J. 1989 Surface waves on shear currents. I. Spectal theory with linear vertical shear. *Rep.* LJI-88-P-486.

JONSSON, I. G. 1989 Wave–current interactions. *Rep.* N. S49. DCAMM, The Technical University of Denmark.

LIGHTHILL, M. J. 1953 On the critical Froude number for turbulent flow over a smooth bottom. *Proc. Camb. Phil. Soc.* **49**, 704–706.

LIN, C. C. 1966 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

LIN, J.-T. & GAD-EL-HAK, M. 1984 Turbulent current measurements in a wind-wave tank. *J. Geophys. Res.* **89**, C1, 627–636.

KIRBY, J. T. & CHEN, T.-M. 1989 Surface waves on vertically sheared flows: approximate dispersion relation. *J. Geophys. Res.* **94**, C1, 1013–1027.

KORN, G. A. & KORN, T. M. 1961 *Mathematical Handbook for Scientists and Engineers*. McGraw-Hill.

MILES, J. 1957 On the generation of surface waves by shear flows. *J. Fluid Mech.* **3**, 185–204 (and Corrigendum **6**, 1959, 582).

MILES, J. 1959 On the generation of surface waves by shear flows. Part 2. *J. Fluid Mech.* **6**, 568–582.

MORLAND, L. C., SAFFMAN, P. G. & YUEN, H. C. 1991 Waves generated by shear flow instabilities. *Proc. R. Soc. Lond.* A **433**, 441–450.

PEREGRINE, D. H. 1976 Interaction of water waves and currents. *Adv. Appl. Mech.* **16**, 9–117.

PEREGRINE, D. H. & JONSSON, I. G. 1983 Interaction of waves and currents. *Misc. Rep.* MR83-6. Coastal Engng Res. Center, US Army Corps of Engrs, Fort Belvoir, Virginia.

SHRIRA, V. I. 1989 On the subsurface waves in the oceanic upper mixed layer. *Dokl. Akad. Nauk. SSSR* **308**, 732–736.

SKOP, R. A. 1987 An approximate dispersion relation for water wave-current interaction. *J. Waterways Port Coastal & Ocean Engng* **113**, 187–195.

STEWART, R. H. & JOY, J. W. 1974 HF radio measurements of surface currents. *Deep-Sea Res.* **21**, 1039–1049.

TRICOMI, F. G. 1970 *Integral Equations*. Interscience.

THOMPSON, P. D. 1949 The propagation of small surface disturbances through rotational flow. *Ann. N.Y. Acad. Sci.* **51**, 463–474.

VORONOVICH, A. G. 1976 Propagation of surface and internal gravity waves in geometric optics approximation. *Atmos. Ocean. Phys.* **12**, 850–867.

WRIGHT, J. & HENYEY, F. 1989 Surface waves on shear currents. II. Spectral theory, critical layer absorption and instability. *Rep.* LJI-89-P-493R1.